Gosford High School

2024

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 1

General Instructions

- Reading time – 10 minutes
- Working time – 2 hours
- Write using black or blue pen
- Calculators approved by NESA may be used
- A laminated reference sheet is provided
- For questions in Section II, show relevant mathematical reasoning and/or calculations

Total Marks: 70

Section I – 10 marks (pages 1 – 4)

- Attempt Questions 1 – 10
- Allow about 15 minutes for this section

Section II – 60 marks (pages 5 – 11)

- Attempt Questions 11 – 14
- Allow about 1 hour and 45 minutes for this section

Section I

10 marks

Attempt Questions 1–10

Allow about 15 minutes for this section.

Use the multiple-choice answer sheet for Questions 1–10.

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1 Evaluate the definite integral $\int_0^{\frac{1}{3}} \frac{1}{\sqrt{1-9x^2}}\,dx$.

A. $\frac{\pi}{2}$

B. $\frac{\pi}{3}$

C. $\frac{\pi}{6}$

D. $\frac{\pi}{4}$

2 Which of the following vectors are perpendicular?

A. $\underset{\sim}{u} = 3\underset{\sim}{i} - 7\underset{\sim}{j}, \quad \underset{\sim}{v} = \frac{7}{3}\underset{\sim}{i} - \underset{\sim}{j}$

B. $\underset{\sim}{u} = \frac{2}{5}\underset{\sim}{i} - 2\underset{\sim}{j}, \quad \underset{\sim}{v} = 5\underset{\sim}{i} + 3\underset{\sim}{j}$

C. $\underset{\sim}{u} = \frac{9}{2}\underset{\sim}{i} + \frac{7}{2}\underset{\sim}{j}, \quad \underset{\sim}{v} = -9\underset{\sim}{i} - \frac{1}{7}\underset{\sim}{j}$

D. $\underset{\sim}{u} = 14\underset{\sim}{i} + \underset{\sim}{j}, \quad \underset{\sim}{v} = \frac{1}{2}\underset{\sim}{i} - 7\underset{\sim}{j}$

3 Consider the differential equation $\frac{dy}{dx} = xy$

Which of the following equations best represents this relationship between x and y ?

A. $y = Ae^{\frac{x^2}{2}}$

B. $y = Ae^x$

C. $y = \log_e |x| + c$

D. $y = \log_e \left| \frac{x^2}{2} + c \right|$

4 The acute angle between the vectors $\underset{\sim}{u} = \begin{pmatrix} \sqrt{5} \\ 3 \end{pmatrix}$ and $\underset{\sim}{v} = \begin{pmatrix} 6 \\ \sqrt{14} \end{pmatrix}$ is (to the nearest degree):

A. 21°

B. 88°

C. 159°

D. 92°

5 If $3\sin x + \sqrt{3}\cos x$ was expressed in the form $A\sin(x+\alpha)$, then the values of A and α would be:

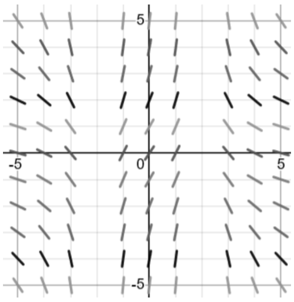
A. $A = 2,\ \alpha = \frac{\pi}{6}$

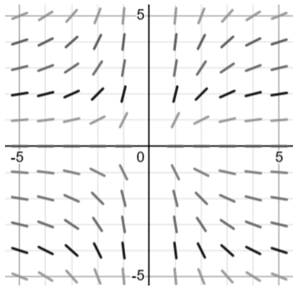
B. $A = 2,\ \alpha = \frac{\pi}{3}$

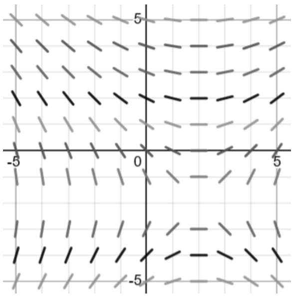
C. $A = 2\sqrt{3},\ \alpha = \frac{\pi}{6}$

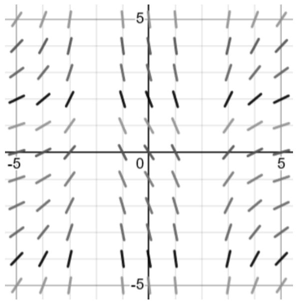
D. $A = 2\sqrt{3},\ \alpha = \frac{\pi}{3}$

6 Which of the following slope fields represents $\dfrac{dy}{dx} = \dfrac{y^2+6}{x^2-4}$?

A.


B.


C.


D.


7 8 adults and 4 children need to be seated at a circular table. How many arrangements exist if the children must sit together?

A. 8709120

B. 967680

C. 362880

D. 120960

8 What is the term independent of x in the expansion of $\left(x^3 + \dfrac{3}{x^2}\right)^{20}$?

A. $\dbinom{20}{8}3^8$

B. $\dbinom{20}{8}3^{12}$

C. $\dbinom{20}{12}3^8$

D. $\dbinom{20}{14}3^6$

9 Find the exact value of the area under the curve $f(x) = \sin^2(x) + 1$ between $x = 0$ and $x = \dfrac{\pi}{12}$.

A. $\dfrac{\pi - 3\sqrt{3}}{8}$ units^2

B. $\dfrac{\pi - 1}{4}$ units^2

C. $\dfrac{\pi - 1}{8}$ units^2

D. $\dfrac{\pi - 3\sqrt{3}}{4}$ units^2

10 When the polynomial $p(x)$ is divided by $x^2 - x - 2$, the remainder is $-\dfrac{1}{3}x + \dfrac{11}{3}$. What is the remainder when $p(x)$ is divided by $x - 2$?

A. 3

B. –3

C. 4

D. –4

Section II

60 marks

Attempt Questions 11–14

Allow about 1 hour and 45 minutes for this section.

Answer each question in the appropriate writing booklet. Extra writing booklets are available.

For questions in Section II, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (16 marks) Use the Question 11 Writing Booklet

(a) Consider the following vectors $\underset{\sim}{u} = 2\underset{\sim}{i} + 5\underset{\sim}{j}$ and $\underset{\sim}{v} = 7\underset{\sim}{i} - 8\underset{\sim}{j}$. Find $\text{proj}_{\underset{\sim}{u}}\, \underset{\sim}{v}$. **2**

(b) Solve the following inequality $\dfrac{2x+1}{x-2} \geq 1$ **3**

(c) Show that $\cos\left[2\tan^{-1}\left(\dfrac{3}{\sqrt{55}}\right)\right] = \dfrac{23}{32}$ **3**

(d) Use the substitution $x = u - 2$ to evaluate $\displaystyle\int_{-1}^{2} \frac{3x+5}{\sqrt{x+2}}\, dx$ **3**

(e) Given polynomial $4x^3 + 3x^2 - 2x + 5 = 0$ has roots α, β, and γ, then find:

i. $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$ **2**

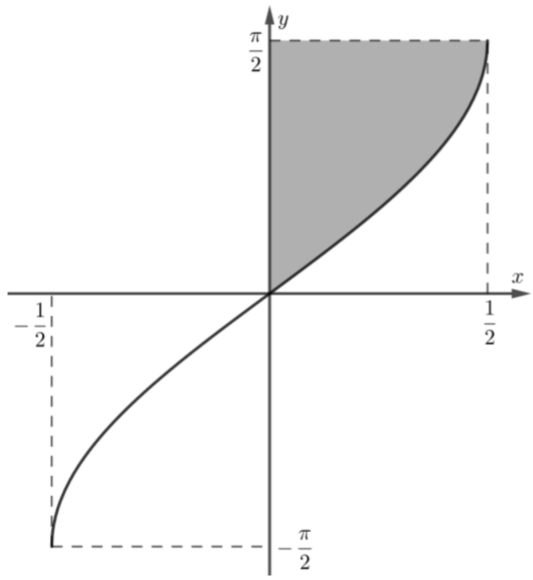
ii. $\alpha^2\beta + \alpha\beta^2 + \alpha\gamma^2 + \alpha^2\gamma + \beta^2\gamma + \beta\gamma^2$ **3**

End of Question 11

Question 12 (16 marks) Use the Question 12 Writing Booklet

(a) Given that $\dfrac{dy}{dx} = \dfrac{1}{1+x^2}$ and $x = 1$ when $y = 0$, find y when $x = \sqrt{3}$. **2**

(b) The diagram below shows the graph of $y = \sin^{-1} 2x$. The shaded area is that bounded by the curve, the y axis and the line $y = \dfrac{\pi}{2}$.



i. Find this shaded area **2**

ii. If this area is rotated about the y axis, find the volume of the solid thus formed. **3**

Question 12 continues on page 7

Question 12 (continued)

(c) The amount Q, measured in milligrams, of a substance present in a chemical reaction at time t minutes is given by the differential equation $\frac{d^2Q}{dt^2} + 6\frac{dQ}{dt} + 9Q = 0$.

i. Show that $Q = (A + Bt)e^{-3t}$ satisfies the differential equation. **2**

ii. Find A and B if Q is 2 initially and Q is $102e^{-15}$ after 5 min. **2**

iii. Find the maximum value of Q and the time at which it occurs. **3**

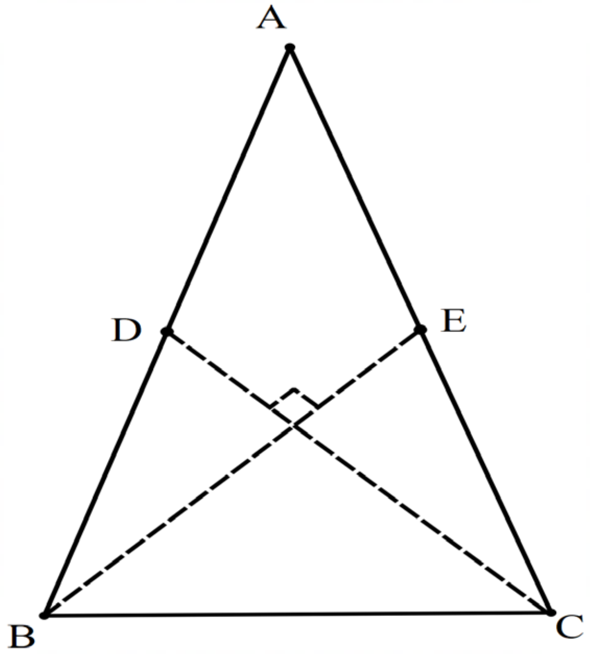
(d) Show that $\dfrac{\sin 2x}{1+\cos 2x} = \tan x$ **2**

End of Question 12

Question 13 (13 marks) Use the Question 13 Writing Booklet

(a) Prove by Mathematical Induction that $7^n + 2$ is divisible by 3 for all $n \in \mathbf{Z}^+$. **3**

(b) In the isosceles triangle ABC, $|\overrightarrow{AB}| = |\overrightarrow{AC}|$. D is the midpoint of side AB and E is the midpoint of side AC. $\overrightarrow{CD}$ is perpendicular to $\overrightarrow{BE}$. **3**



Use **vector methods** to find the size of $\angle BAC$, correct to the nearest degree.

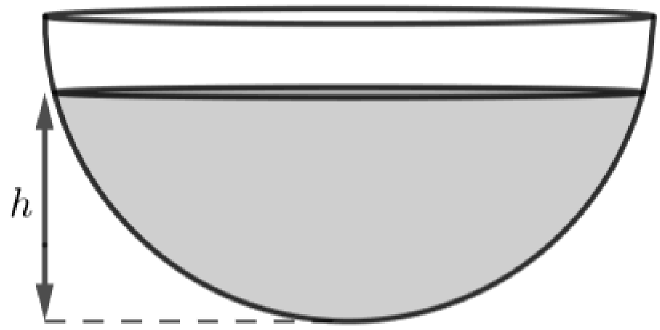
(c) Consider the expansion of $(1+x)^{n-1}$ for integers $n > 2$.

Show that $n\binom{n-1}{1} + n\binom{n-1}{2} + \ldots + n\binom{n-1}{n-2} = n\left(2^{n-1} - 2\right)$ **2**

Question 13 continues on page 9

Question 13 (continued)

(d) A hollow sphere with a diameter of 20 m is cut in half and its top removed. Water is poured into the remaining half of the sphere at a constant rate of 20 $\mathrm{m}^3/\mathrm{minute}$. After 1 minute, the water level is h cm above the base of the semi sphere.



i. Show that an expression for the volume of the water in terms of h is: **3**

$$V = \frac{1}{3}\pi h^2 (30 - h)$$

Hint: consider the volume of rotation.

ii. Find the rate of change of the water level when the water is 6 m above the base. **2**

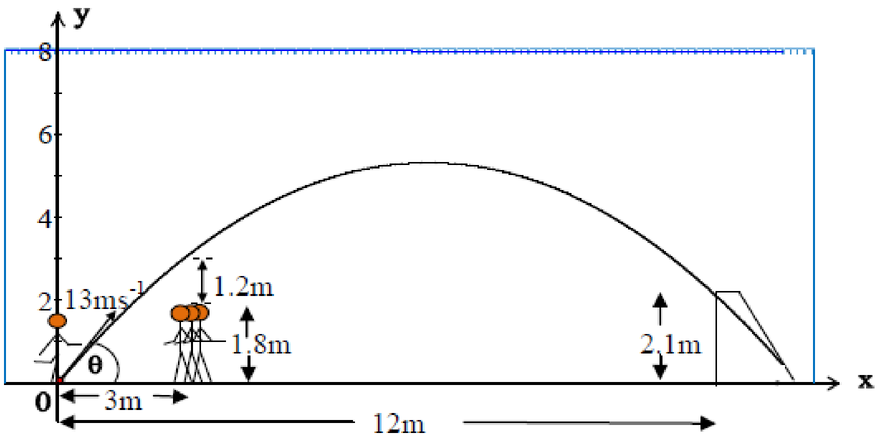
End of Question 13

Question 14 (15 marks) Use the Question 14 Writing Booklet

(a) Solve $\cos^2 x + \sin x - 1 = 0$ for $0 \le x \le 2\pi$. **3**

(b) Joe is playing indoor soccer. He is to take a free kick from the origin. The opposing team has positioned some of its players $3\,\text{m}$ from the ball. These players are $1.8\,\text{m}$ tall but can jump an extra $1.2\,\text{m}$ to defend their goal. The ceiling of the stadium is $8\,\text{m}$ above the floor. Joe will kick the ball with a velocity of $13\,\text{ms}^{-1}$ at an angle of θ to the horizontal. Use the axes as shown, and assume there is no air resistance, $g = 10\,\text{ms}^{-2}$ and the position of Joe's ball t seconds after being kicked is given by the equations:

$x = 13t\cos\theta$ and $y = 13t\sin\theta - 5t^2$ (Do NOT prove this).



i. Show that the maximum height reached by the ball can be expressed as **2**

$$h = \frac{169\sin^2\theta}{20}.$$

ii. Show that the cartesian equation for the trajectory of the ball is **2**

$$y = x\tan\theta - \frac{5x^2}{169}\left(1+\tan^2\theta\right).$$

iii. Show that for the ball to pass over the defenders and below the ceiling that angle θ must be between $50°41'$ and $76°39'$. **3**

Question 14 continues on page 11

Question 14 (continued)

(c) A tiny pond is initially unpolluted and holds 450 litres of water. There is an input stream with a concentration of $5\,\text{g/L}$ of pollutant flowing into the pond at a rate of $30\,\text{L/day}$. Moreover, there is an output stream with a volumetric flow rate of $30\,\text{L/day}$. Let m be the mass (in grams) of pollutant in the pond and t be the number of days after the input stream first begins to flow into the pond.

i. Show that the rate of change of the mass of the pollutant, m, after t days is: **2**

$$\frac{dm}{dt} = 150 - \frac{m}{15}$$

ii. What is the concentration of the pollutant in the pond after 10 days? **3**

End of Paper

GHS 2024 E1 Multiple Choice Worked Solutions

No	Working	Answer
1	$\int_0^{\frac{1}{3}} \frac{1}{\sqrt{1-9x^2}}\,dx = \frac{1}{3}\int_0^{\frac{1}{3}} \frac{3}{\sqrt{(1)^2-(3x)^2}}\,dx$ $= \frac{1}{3}\left[\sin^{-1}(3x)\right]_0^{\frac{1}{3}}$ $= \frac{\pi}{6}$	**C**
2	The perpendicular vectors have a zero dot product. $\underset{\sim}{u}\cdot\underset{\sim}{v} = \left(14\,\underset{\sim}{i} + \underset{\sim}{j}\right)\cdot\left(\frac{1}{2}\underset{\sim}{i} - 7\,\underset{\sim}{j}\right)$ $= \left(14\cdot\frac{1}{2}\right) + (1\cdot -7)$ $= 7 - 7 = 0$	**D**
3	$\frac{dy}{dx} = xy$ $\frac{1}{y}\times\frac{dy}{dx} = x$ $\int \frac{1}{y}\,dy = \int x\,dx$ $\ln y = \frac{x^2}{2} + c$ $y = Ae^{\frac{x^2}{2}}$	**A**
4	$\cos^{-1}\left(\frac{u\cdot v}{\lvert u\rvert\lvert v\rvert}\right) = \cos^{-1}\left(\frac{6\sqrt{5}+3\sqrt{14}}{\sqrt{14}\cdot\sqrt{50}}\right)$ $\approx 21°$	**A**
5	$A = (a)^2 + (b)^2$: $A^2 = 12 \rightarrow A = 2\sqrt{3}$ $\tan\alpha = \frac{1}{\sqrt{3}}$ $\alpha = \frac{\pi}{6}$	**C**
6	The slope field is $\frac{dy}{dx} = \frac{y^2+6}{x^2-4} = \frac{y^2+6}{(x-2)(x+2)}$. So, the slope field is not defined at $x = \pm 2$ and has vertical asymptotes at these points. Next, $x^2 - 4 > 0$ for $x < -2$ or $x > 2$, and $x^2 - 4 < 0$ for $-2 < x < 2$. Therefore, the slope field has positive gradients on $x < -2$ or $x > 2$, and has negative gradients on $-2 < x < 2$.	**D**
7	First, consider the four children as a bundle so that with the other 8 adults there are initially 9 entities to be arranged in a circle. The number of ways to arrange 7 entities in a circle is $(9-1)! = 8!$. Then, the number of ways to arrange the four children in that bundle (in a line) is $4!$. By the multiplication rule of counting, the number of arrangements is $8! \times 4! = 967680$	**B**
8	Swap the terms in the expansion around so that it is $\left(\frac{3}{x^2} + x^3\right)^{20}$ Then use the general term $T_{k+1} = \binom{20}{k} 3^{20-k}(x^{-2})^{20-k}(x^3)^k$ to find constant term (x^0). Alternatively, test each option given to see which would work with the expansion.	**B**

9	$$\int_0^{\frac{\pi}{12}} \sin^2(x) + 1\, dx = \int_0^{\frac{\pi}{12}} \left(\frac{1}{2} - \frac{1}{2}\cos(2x)\right) + 1\, dx$$ $$= \int_0^{\frac{\pi}{12}} \frac{3}{2} - \frac{1}{2}\cos(2x)\ dx$$ $$= \left[\frac{3x}{2} - \frac{\sin(2x)}{4}\right]_0^{\frac{\pi}{12}}$$ $$= \left(\frac{3\pi}{24} - \frac{\sin\left(\frac{2\pi}{12}\right)}{4}\right) - (0 - 0)$$ $$= \frac{\pi}{8} - \frac{\sin\left(\frac{\pi}{6}\right)}{4}$$ $$= \frac{\pi}{8} - \frac{1}{8}$$ $$= \frac{\pi - 1}{8}\ \text{units}^2$$	**C**
10	From the remainder theorem, $p(-1) = 4$. Noting that $x^2 - x - 2 = (x-2)(x+1)$, write: $$p(x) = (x-2)(x+1)Q(x) + \left(-\frac{1}{3}x + \frac{11}{3}\right)$$ $p(2)$: $R(2) = -\frac{1}{3}(2) + \frac{11}{3}$ $= 3$ So, the remainder is 3	**A**

Markers Comments

Trial HSC Examination
2024

Mathematics Extension 1

Name ______________________ Teacher ____________________

Section I – Multiple Choice Answer Sheet

Allow about 15 minutes for this section

Select the alternative A, B, C or D that best answers the question. Fill in the response oval completely.

Sample:	2 + 4 =	(A) 2	(B) 6	(C) 8	(D) 9
		A ○	B ●	C ○	D ○

If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.

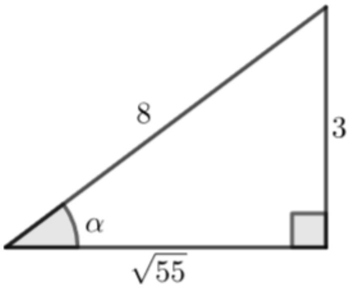
A ● B ~~●~~ C ○ D ○

If you change your mind and have crossed out what you consider to be the correct answer, then indicate the correct answer by writing the word **correct** and drawing an arrow as follows.

correct

A ~~●~~ B ~~●~~ C ○ D ○

1.	A ○	B ○	C ●	D ○
2.	A ○	B ○	C ○	D ●
3.	A ●	B ○	C ○	D ○
4.	A ●	B ○	C ○	D ○
5.	A ○	B ○	C ●	D ○
6.	A ○	B ○	C ○	D ●
7.	A ○	B ●	C ○	D ○
8.	A ○	B ●	C ○	D ○
9.	A ○	B ○	C ●	D ○
10.	A ●	B ○	C ○	D ○

11		GHS Ext 1 HSC 2024 Question 11 Worked Solutions (16 marks)	Marks	Allocation and Comments
(a)		$\underset{\sim}{v} \cdot \underset{\sim}{u} = 14 - 40 = -26$ $\underset{\sim}{u} \cdot \underset{\sim}{u} = 2^2 + 5^2 = 29$ So, the projection of v onto u is: $$\text{proj}_{\underset{\sim}{u}}\, \underset{\sim}{v} = \left(\frac{\underset{\sim}{v} \cdot \underset{\sim}{u}}{\underset{\sim}{u} \cdot \underset{\sim}{u}}\right)\underset{\sim}{u} = -\frac{26}{29}\left(2\,\underset{\sim}{i} + 5\,\underset{\sim}{j}\right)$$	2	**2 for correct solution** **1 for correct dot product or some correct progress.**
(b)		$$\frac{2x+1}{x-2} \geq 1$$ $$(x-2)(2x+1) \geq (x-2)^2$$ $$(x-2)(2x+1) - (x-2)^2 \geq 0$$ $$(x-2)(2x+1-x+2) \geq 0$$ $$(x-2)(x+3) \geq 0$$ $$\therefore\ x \leq -3, \qquad x > 2$$	3	**3 for correct solution** **2 for significant correct progress.** **1 for some correct progress towards solution**
(c)		$\text{LHS} = \cos\left[2\tan^{-1}\left(\frac{3}{\sqrt{55}}\right)\right]$ Let $\alpha = \tan^{-1}\left(\frac{3}{\sqrt{55}}\right)$ so that $\tan\alpha = \frac{3}{\sqrt{55}}$ which corresponds to the following triangle:  From the triangle above, we have $\cos\alpha = \frac{\sqrt{55}}{8}$ and $\sin\alpha = \frac{3}{8}$. $\text{LHS} = \cos\left[2\tan^{-1}\left(\frac{3}{\sqrt{55}}\right)\right]$ $= \cos 2\alpha$ $= \cos^2\alpha - \sin^2\alpha$ $= \left(\frac{\sqrt{55}}{8}\right)^2 - \left(\frac{3}{8}\right)^2$ $= \frac{23}{32}$ $= \text{RHS}$ as required.	3	**3 for correct solution** **2 for applying the double angle formula for cosine.** **1 for finding the correct values of $\cos\left[\tan^{-1}\left(\frac{3}{\sqrt{55}}\right)\right]$ and $\sin\left[\tan^{-1}\left(\frac{3}{\sqrt{55}}\right)\right]$ or for drawing correct triangle or equivalent measure.**

<table>
<tr><td colspan="2">11</td><td>GHS Ext 1 HSC 2024
Question 11 Worked Solutions (16 marks)</td><td>Marks</td><td>Allocation and Comments</td></tr>
<tr><td>(d)</td><td></td><td>Using substitution:
$\frac{dx}{du} = 1$
$\boldsymbol{dx = du}$
$u = x + 2$
Lower bound: $\boldsymbol{u = 1}$, Upper bound: $\boldsymbol{u = 4}$

$\int_{-1}^{2} \frac{3x+5}{\sqrt{x+2}}\, dx = \int_{1}^{4} \frac{3u-1}{\sqrt{u}}\, du$
$= \int_1^4 (3u-1)u^{-\frac{1}{2}}\, du$
$= \int_1^4 3u^{\frac{1}{2}} - u^{-\frac{1}{2}}\, du$
$= \left[2u^{\frac{3}{2}} - 2u^{\frac{1}{2}}\right]_1^4$
$= \left(2\sqrt{4^3} - 2\sqrt{4}\right) - (2-2)$
$= 12$</td><td>3</td><td>3 for correct solution.

2 for making significant progress towards solution with one error.

1 for stating the correct bounds for u or correctly expresses the integral in terms of $\boldsymbol{u}$.</td></tr>
<tr><td>(e)</td><td>(i)</td><td>From the given cubic polynomial, we conjecture that:
$$\alpha\beta + \alpha\gamma + \beta\gamma = -\frac{1}{2}, \alpha\beta\gamma = -\frac{5}{4}$$
Then, it follows:
$$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\alpha\beta + \alpha\gamma + \beta\gamma}{\alpha\beta\gamma}$$
$$= \frac{\left(-\frac{1}{2}\right)}{\left(-\frac{5}{4}\right)} = \frac{2}{5}$$</td><td>2</td><td>2 for correct solution.
1 for some correct progress made</td></tr>
<tr><td></td><td>(ii)</td><td>$(\alpha+\beta+\gamma)(\alpha\beta+\alpha\gamma+\beta\gamma) = \alpha^2\beta + \alpha\beta^2 + \alpha\gamma^2 + \alpha^2\gamma + \beta^2\gamma + \beta\gamma^2 + 3\alpha\beta\gamma$

$\alpha^2\beta + \alpha\beta^2 + \alpha\gamma^2 + \alpha^2\gamma + \beta^2\gamma + \beta\gamma^2$
$= (\alpha+\beta+\gamma)(\alpha\beta+\alpha\gamma+\beta\gamma) - 3\alpha\beta\gamma$
$= -\frac{3}{4}\left(-\frac{1}{2}\right) - 3\left(-\frac{5}{4}\right)$
$= \frac{33}{8}$</td><td>3</td><td>3 for correct solution.
2 for significant progress towards solution.
1 for some correct progress.</td></tr>
</table>

11	GHS Ext 1 HSC 2024 Question 11 Worked Solutions (16 marks)	Marks	Allocation and Comments

Markers Comments

a) Most students answered part a very well. Main error was when students did the projection of u onto v and not v onto u as was asked in the question.
b) Students generally seemed to know what to do though main mistakes were from lazy algebraic errors or from not remembering that x could not equal 2, which is a main feature of this type of inequality.
c) Students who thought to draw a triangle from the given angle were generally able to make a more convincing attempt at this question. Typing show that questions into a calculator to get a non-exact value will **never** be a valid method of answering this type of question and will never be awarded marks.
d) Generally students were able to make a good attempt at this question. Main mistakes occurred when dividing $\sqrt{u}$ into the numerator. Take care when working with index laws.
e) Part i was generally done very well though it shouldn't have to be said that $\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma} \neq \frac{1}{\alpha+\beta+\gamma}$
Part ii was a difficult expression to form. Many students struggled with this. When making a claim that a factorised expression is equal to the question, an expansion should be provided to prove/check.

12		GHS Ext 1 HSC 2024 Question 12 Worked Solutions (16 marks)	Marks	Allocation and Comments
(a)		$\frac{dy}{dx}=\frac{1}{1+x^2}$ $y=\tan^{-1}(x)+c$ $Sub\ in\ (1,0)$ $0=\tan^{-1}1+c$ $c=-\frac{\pi}{4}$ $\therefore y=\tan^{-1}(x)-\frac{\pi}{4}$ $When\ x=\sqrt{3}:$ $y=\tan^{-1}\sqrt{3}-\frac{\pi}{4}$ $y=\frac{\pi}{3}-\frac{\pi}{4}$ $\therefore y=\frac{\pi}{12}$	2	**2 for correct solution.** **1 for correct integration but incorrect evaluation of *y*.** ***Marker's Comments*** *Generally well answered.* *Poorer responses incorrectly integrated or made careless errors in evaluating the constant or value for y.* Tips for improvement. • Use the reference sheet to identify the correct form of the integral. • Show more steps to avoid arithmetic errors.

<table>
<tr><td colspan="2">12</td><td>GHS Ext 1 HSC 2024
Question 12 Worked Solutions (16 marks)</td><td>Marks</td><td>Allocation and Comments</td></tr>
<tr><td>(b)</td><td>(i)</td><td>$$\int_0^{\frac{\pi}{2}} \frac{1}{2} sin\, y\ \ dy = \left[-\frac{1}{2} cos\, y\right]_0^{\frac{\pi}{2}}$$ $$= -\frac{1}{2}(0-1)$$ $$= \frac{1}{2} u^2$$</td><td>2</td><td>2 for correct solution.
1 for writing the integral that represents the shaded area.
Marker’s Comments
Generally well answered.
Poorer responses were unable to write the integral in terms of y.

Tips for improvement.
• Review questions that ask to find the area between a curve and the y-axis.</td></tr>
</table>

12		GHS Ext 1 HSC 2024 Question 12 Worked Solutions (16 marks)	Marks	Allocation and Comments
	(ii)	$$\pi\int_0^{\frac{\pi}{2}}\left(\frac{1}{2}\sin y\right)^2 dy = \frac{\pi}{4}\int_0^{\frac{\pi}{2}}\frac{1}{2}(1-\cos 2y)\,dy$$ $$= \frac{\pi}{8}\left[y-\frac{1}{2}\sin 2y\right]_0^{\frac{\pi}{2}}$$ $$= \frac{\pi}{8}\left(\frac{\pi}{2}\right)$$ $$= \frac{\pi^2}{16}$$	3	**3 for correct solution.** **2 for integrating $\int_0^{\frac{\pi}{2}}\left(\frac{1}{2}\sin y\right)^2 dy$ and obtaining a result involving a trigonometric ratio.** **1 for finding an integral expression of the volume.** ***Marker's Comments*** *Many students experienced difficulty with this question.* *Poorer responses incorrectly calculated the constant to be factored outside of the integral or found the volume around the x-axis* Tips for improvement. • Show more working out by (1)Taking time to consider the axis in which the function is being rotated, (2) write out the formula to find the volume, (in this case) $V=\pi\int x^2\ dy$ (3) make (in this case) x^2 the subject – you should have an expression in terms of *y* (4) use an identity to rewrite the expression in order to integrate

12		GHS Ext 1 HSC 2024 Question 12 Worked Solutions (16 marks)	Marks	Allocation and Comments
(c)	(i)	First, find the first and second derivatives of Q $$\frac{dQ}{dt} = (-3A + B - 3Bt)e^{-3t}$$ $$\frac{d^2Q}{dt^2} = (9A - 6B + 9Bt)e^{-3t}$$ Substitute the function Q and its derivatives into the differential equation to get: $\frac{d^2Q}{dt^2} + 6\frac{dQ}{dt} + 9Q$ $= (9A - 6B + 9Bt)e^{-3t} + 6(-3A + B - 3Bt)e^{-3t} + 9(A + Bt)e^{-3t}$ $= [(9A - 18A + 9A) + (-6B + 6B) + (9B - 18B + 9B)t]e^{-3t}$ $= 0\ (shown)$	2	**2 for correct solution.** **1 for finding $\frac{d^2Q}{dt^2}$ and $\frac{dQ}{dt}$** ***Marker's Comments*** *Most students could not differentiate correctly.* *The derivatives involved use of the product rule.* Tips for improvement. • Ask yourself what are you deriving with respect to. Consider whether you have to use a rule.
	(ii)	First condition: $Q(0) = 2$ $$[A + (B \cdot 0)]e^{-3(0)} = 2$$ $$\therefore A = 2$$ Second condition: $Q(5) = 102e^{-15}$ $$(A + 5B)e^{-15} = 102e^{-15}$$ $$2 + 5B = 102$$ $$\therefore B = 20$$	2	**2 for correct solution.** **1 for obtaining one of the correct value for either A or B.** ***Marker's Comments*** *Generally well answered.* *Poorer responses made careless errors in evaluating the values for A and B.* Tips for improvement. • Space your work out to avoid needless errors.

12	GHS Ext 1 HSC 2024 Question 12 Worked Solutions (16 marks)	Marks	Allocation and Comments

(iii)

Marks: 3

$$\frac{dQ}{dt} = (14 - 60t)e^{-3t}$$
$$(14 - 60t)e^{-3t} = 0$$
$$t = \frac{7}{30}$$

Proving that there is a relative maximum at the critical point above using a table of values:

t	0	$\frac{7}{30}$	1
$\frac{dQ}{dt}$	$14 > 0$	0	$-\frac{46}{e^3} < 0$
	$\nearrow$	$\rightarrow$	$\searrow$

The maximum value of Q is:

$$Q = \left(2 + 20 \cdot \frac{7}{30}\right) e^{-3\left(\frac{7}{30}\right)} = \frac{20}{3} e^{-\frac{7}{10}}$$

3 for correct solution.

2 for incorrect answer but testing for a maximum
1 for finding the value of t for within Q is maximised or incorrect value for the *t* based on CTE.

Marker's Comments

Poorly answered.

Poorer responses found negative values for t and did not question why they were getting a negative term.

Many students did not test for a maximum.

Tips for improvement.

- Errors such as a negative value for *t* typically indicate an error has been made with the previous working. Students should then check previous parts for an error.

(d)

Marks: 2

$$LHS = \frac{\sin 2x}{1 + \cos 2x}$$
$$= \frac{2 \sin x \cos x}{1 + \cos^2 x - \sin^2 x}$$
$$= \frac{2 \sin x \cos x}{1 + \cos^2 x - (1 - \cos^2 x)}$$
$$= \frac{2 \sin x \cos x}{2 \cos^2 x}$$
$$= \frac{\sin x}{\cos x}$$
$$= \tan x$$
$$= RHS$$

2 for correct solution with no steps omitted.

1 for correct progress.

Marker's Comments

Generally well answered.

Poorer responses skipped steps in the working.

Tips for improvement.

- In "show that" questions, students should be more explicit, showing all steps no matter how "trivial" they feel the step may be.

12	GHS Ext 1 HSC 2024 **Question 12 Worked Solutions (16 marks)**	Marks	**Allocation and Comments**

Markers Comments

13		**GHS Ext 1 HSC 2024** **Question 13 Worked Solutions (13 marks)**	Marks	**Allocation and Comments**
(a)		Initial Step: When $n = 1$, $7^1 + 2 = 9$ is divisible by 3. So, it is true for $n = 1$ Assumption: Assume true for $n = k$ $(k \in \mathbb{Z}^+)$: $7^k + 2 = 3M, \quad M \in \mathbb{Z} \quad (*)$ Inductive Step: RTP: Prove to be true for $n = k + 1$: (**Note:** rearrange the equation $(*)$ to get $7^k = 3M - 2$) $7^{k+1} + 2 = 7 \cdot 7^k + 2$ $= 7(3M - 2) + 2$ $= 21M - 14 + 2$ $= 21M - 12$ $= 3(7M - 4)$ $= 3Q,\ Q \in \mathbb{Z}$ So, it is true for $n = k + 1$ Therefore true for all positive integers n by the principles of Mathematical Induction.	3	**3 for correct solution including conclusion.** **2 for proving that if the statement is true for n=k, then it must be true for n=k+1.** **1 for verifying the base case.**

Markers Comments

Generally well done by most students. They knew how to verify the initial case and correctly substitute the rearranged assumption. It was good to see they acknowledged that their pronumeral was an integer.

This question was a straightforward application of Induction.

13		GHS Ext 1 HSC 2024 Question 13 Worked Solutions (13 marks)	Marks	Allocation and Comments
(b)		Let $\overrightarrow{AB} = \boldsymbol{a}$ and let $\overrightarrow{AC} = \boldsymbol{b}$. Then $\lvert\boldsymbol{a}\rvert = \lvert\boldsymbol{b}\rvert$. $\overrightarrow{CD} = \overrightarrow{CA} + \frac{1}{2}\overrightarrow{AB}$ $\overrightarrow{CD} = \frac{1}{2}\overrightarrow{AB} - \overrightarrow{AC}$ $\overrightarrow{CD} = \frac{1}{2}\boldsymbol{a} - \boldsymbol{b}$ $\overrightarrow{BE} = \overrightarrow{BA} + \frac{1}{2}\overrightarrow{AC}$ $\overrightarrow{BE} = \frac{1}{2}\overrightarrow{AC} - \overrightarrow{AB}$ $\overrightarrow{BE} = \frac{1}{2}\boldsymbol{b} - \boldsymbol{a}$ $\overrightarrow{CD} \cdot \overrightarrow{BE} = 0$ (Vectors are perpendicular). $\left(\frac{1}{2}\boldsymbol{a} - \boldsymbol{b}\right)\left(\frac{1}{2}\boldsymbol{b} - \boldsymbol{a}\right) = 0$ $\frac{1}{4}\boldsymbol{a}\cdot\boldsymbol{b} - \frac{1}{2}\boldsymbol{a}\cdot\boldsymbol{a} - \frac{1}{2}\boldsymbol{b}\cdot\boldsymbol{b} + \boldsymbol{a}\cdot\boldsymbol{b} = 0$ ($\boldsymbol{a}\cdot\boldsymbol{a} = \lvert\boldsymbol{a}\rvert^2$ and $\boldsymbol{b}\cdot\boldsymbol{b} = \lvert\boldsymbol{b}\rvert^2$) $\frac{5}{4}\boldsymbol{a}\cdot\boldsymbol{b} - \frac{1}{2}\lvert\boldsymbol{a}\rvert^2 - \frac{1}{2}\lvert\boldsymbol{b}\rvert^2 = 0$ ($\lvert\boldsymbol{a}\rvert = \lvert\boldsymbol{b}\rvert$) $\frac{5}{4}\boldsymbol{a}\cdot\boldsymbol{b} - \lvert\boldsymbol{a}\rvert^2 = 0$ $\frac{5}{4}\lvert\boldsymbol{a}\rvert\lvert\boldsymbol{b}\rvert\cos(\angle BAF) - \lvert\boldsymbol{a}\rvert^2 = 0$ ($\lvert\boldsymbol{a}\rvert = \lvert\boldsymbol{b}\rvert$) $\frac{5}{4}\lvert\boldsymbol{a}\rvert^2\cos(\angle BAF) - \lvert\boldsymbol{a}\rvert^2 = 0$ $\frac{5}{4}\cos(\angle BAF) - 1 = 0$ $\frac{5}{4}\cos(\angle BAF) = 1$ $\cos(\angle BAF) = \frac{4}{5}$ $\angle BAF = \cos^{-1}\left(\frac{4}{5}\right)$ $\angle BAF \approx 36°52'$ (nearest minute)	3	**3 for correct proof.** **2 for utilising angle between vectors or equivalent progress.** **1 for utilising expansion of dot product or equivalent progress.**

Markers Comments

Was not well done. Many students did not know how to start the question by letting the equal sides of the isosceles triangle be two different vectors. Then getting expressions for CD and BE.

Better responses used the sides AB and AC rather than BC. Suggest you try and limit the number of vectors you define to 2 rather than 3.

Students need to be careful with direction of vectors as they used positive vectors for both directions rather than making the opposite direction negative.

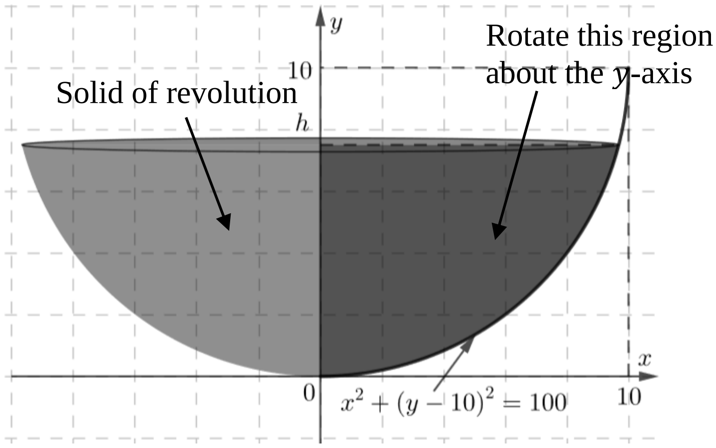
Most responses knew they needed to use the dot product and given a right angle, the dot product should be equal to 0.

13		GHS Ext 1 HSC 2024 **Question 13 Worked Solutions (13 marks)**	Marks	**Allocation and Comments**
(c)		$(1+x)^{n-1}=1+\binom{n-1}{1}x+\binom{n-1}{2}x^2+\ldots+\binom{n-1}{n-1}x^{n-1}+x^{n-1}$ Let $x=1$: $(1+1)^{n-1}=1+\binom{n-1}{1}+\binom{n-1}{2}+\ldots+\binom{n-1}{n-2}+1$ $2^{n-1}=2+\binom{n-1}{1}+\binom{n-1}{2}+\ldots+\binom{n-1}{n-2}$ $2^{n-1}-2=\binom{n-1}{1}+\binom{n-1}{2}+\ldots+\binom{n-1}{n-2}$ $n\left(2^{n-1}-2\right)=n\left[\binom{n-1}{1}+\binom{n-1}{2}+\ldots+\binom{n-1}{n-2}\right]$ $\therefore n\binom{n-1}{1}+n\binom{n-1}{2}+\ldots+n\binom{n-1}{n-2}=n\left(2^{n-1}-2\right)$ as required	2	**2 for correct solution** **1 for progress towards correct answers**

Markers Comments

Many students attempted to expand the binomial, but made careless errors.

Better responses realised they needed to substitute in a value of 1 for x to get the number 2.

<table>
<tr><td colspan="2">13</td><td>GHS Ext 1 HSC 2024
Question 13 Worked Solutions (13 marks)</td><td>Marks</td><td>Allocation and Comments</td></tr>
<tr><td>(d)</td><td>(i)</td><td>Note that the required volume is the volume of revolution of part of a circle centred at $(0, 10)$ with radius 10, given by $x^2 + (y - 10)^2 = 100$, over the interval $0 \leq y \leq h$ about the y-axis.


Rearrange the equation of the circle into:
$x^2 = 100 - (y - 10)^2$
$x^2 = 20y - y^2$
Then, the required volume V of revolution is given by:
$V = \pi \int_0^h x^2 \, dy$
$= \pi \int_0^h (20y - y^2) \, dy$
$= \pi \left[10y^2 - \frac{1}{3} y^3 \right]_0^h$
$= \frac{1}{3} \pi h^2 (30 - h)$</td><td>3</td><td>3 for correct solution.

2 for correct integral or equivalent progress.

1 for considering volume of revolution or equivalent method.</td></tr>
<tr><td></td><td>(ii)</td><td>Differentiate the expression of volume found in part (c) (i) with respect to h:
$\frac{dV}{dh} = \pi \frac{d}{dh} \left(10h^2 - \frac{1}{3} h^3 \right)$
$\frac{dV}{dh} = \pi (20h - h^2)$
When $h = 6$, we have:
$\frac{dV}{dh} = \pi (20 \cdot 6 - 6^2) = 84\pi$
Given that $\frac{dV}{dt} = 20$ and by using the chain rule, it follows:
$\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$
$20 = 84\pi \times \frac{dh}{dt}$
$\therefore \frac{dh}{dt} = \frac{5}{21\pi} m/minute$</td><td>2</td><td>2 for correct solution

1 for correct expression for $\frac{dV}{dh}$.</td></tr>
</table>

13	**GHS Ext 1 HSC 2024** **Question 13 Worked Solutions (13 marks)**	Marks	**Allocation and Comments**

Markers Comments

Due to an error in the question, this question was removed.

14		GHS Ext 1 HSC 2024 **Question 14 Worked Solutions (15 marks)**	Marks	**Allocation and Comments**
(a)		$\cos^2 x + \sin x - 1 = 0$ $1 - \sin^2 x + \sin x - 1 = 0$ $\sin x - \sin^2 x = 0$ $\sin x(1 - \sin x) = 0$ $\sin x = 0 \qquad \sin x = 1$ $\therefore x = 0, \frac{\pi}{2}, \pi \text{ and } 2\pi$	3	**3 for correct solution** **2 for forming a quadratic in sine or significant progress.** **1 for making some progress towards solving the equation.**
Mixed results. Many students did not recognise or correctly apply the Pythagorean identity. A common error was to try to write a double angle formula or missing a result.				
(b)	(i)	i) $y = 13t\sin\theta - 5t^2$ So $\dot{y} = 13\sin\theta - 10t$ To find the time taken by the ball to reach the maximum height we let $\dot{y} = 0$, we get: $13\sin\theta - 10t = 0$ $13\sin\theta = 10t$ $t = \frac{13}{10}\sin\theta$ Now, the maximum height is $y_{max} = 13 \times \frac{13}{10}\sin\theta \times \sin\theta - 5 \times \frac{169}{100}\sin^2\theta$ $= \frac{169}{10}\sin^2\theta - \frac{169}{20}\sin^2\theta$ $= \frac{169}{20}\sin^2\theta$	2	**2 for correct solution.** **1 for some correct progress.**
	(ii)	ii) $x = 13t\cos\theta \quad \text{so} \quad t = \frac{x}{13\cos\theta}$ As $y = 13t\sin\theta - 5t^2$ then $y = \frac{13\sin\theta\, x}{13\cos\theta} - 5 \times \frac{x^2}{169\cos^2\theta}$ $y = x\tan\theta - \frac{5x^2}{169}\sec^2\theta$ Hence $y = x\tan\theta - \frac{5x^2}{169}(1 + \tan^2\theta)$	2	**2 for correct solution.** **1 for some correct progress.**

14	GHS Ext 1 HSC 2024 Question 14 Worked Solutions (15 marks)	Marks	Allocation and Comments
(iii)	iii) For the ball to pass under the ceiling, if the maximum height $y_{max} < 8$ that is $\frac{169}{20}\sin^2\theta < 8$ $\sin^2\theta < \frac{160}{169}$ $\frac{-4\sqrt{10}}{13} < \sin\theta < \frac{4\sqrt{10}}{13}$ But θ is an acute angle so $\sin\theta > 0$ that is $0 < \sin\theta < \frac{4\sqrt{10}}{13}$ Now as sin θ is an increasing function for θ between 0° and 90° then $0^0 < \theta < 76°\,39'$ (A) The ball must not be stopped by the defenders. This occur if when $x = 3$, $y > 3$ that is $3\tan\theta - \frac{45}{169}(1 + \tan^2\theta) > 3$ $507\tan\theta - 45 - 45\tan^2\theta - 507 > 0$ $45\tan^2\theta - 507\tan\theta + 552 < 0$ First we solve $45\tan^2\theta - 507\tan\theta + 552 = 0$, we get: $\tan\theta = \frac{507 \pm \sqrt{(-507)^2 - 4 \times 45 \times 552}}{90}$ $= 1.22110\ldots, 10.04556..$ As $y = 45\tan^2\theta - 507\tan\theta + 552$ is a parabola below the tan θ axis when tanθ is between 1.22110… and 10.04556… So, the solution of $45\tan^2\theta - 507\tan\theta + 552 < 0$ is $1.22110\ldots < \tan\theta < 10.04556\ldots$ As tan θ is an increasing function for θ between 0° and 90° then $50°41' < \theta < 84°19'$ (B) Now, for the ball to pass over the defenders, but under the ceiling, it must satisfy both conditions (A) and (B) Hence $50°41' < \theta < 76°\,39'$ (1)	3	**3 for correct solution.** **2 for finding both restrictions for theta but no combining them to give the desired restriction or significant progress with one mistake.** **1 for calculating solution for angle needed to overcome the defenders.**

Part (i)and (ii) were mostly well done. (iii) was poorly done by most of the cohort. Many one found one angle result.

14		GHS Ext 1 HSC 2024 **Question 14 Worked Solutions (15 marks)**	Marks	**Allocation and Comments**
(c)	(i)	$$\frac{dm}{dt} = (5 \cdot 30) - 30\left(\frac{m}{450}\right)$$ $$\therefore \frac{dm}{dt} = 150 - \frac{1}{15}m, m(0) = 0$$	2	**2 for correctly constructing the differential equation.** **1 for some correct progress.**

Mostly well done. You must show all steps!

14		GHS Ext 1 HSC 2024 Question 14 Worked Solutions (15 marks)	Marks	Allocation and Comments
	(ii)	$$\frac{dm}{dt} = 150 - \frac{1}{15}m$$ $$\frac{1}{150 - \frac{m}{15}}dm = dt$$ $$\int \frac{1}{150 - \frac{m}{15}}dm = \int dt$$ $$-15\,ln\left\|150 - \frac{1}{15}m\right\| = t + C$$ With the initial condition $m(0) = 0$, we have: $$C = -15\,ln\,150$$ Then, the particular solution of the differential equation is: $$-15\,ln\left\|150 - \frac{1}{15}m\right\| = t - 15\,ln\,150$$ $$15\,ln\,150 - 15\,ln\left\|150 - \frac{1}{15}m\right\| = t$$ $$15\,ln\left\|\frac{150}{150 - \frac{m}{15}}\right\| = t$$ $$ln\left\|\frac{150}{150 - \frac{m}{15}}\right\| = \frac{1}{15}t$$ $$\frac{150}{150 - \frac{m}{15}} = e^{\frac{1}{15}t}$$ $$150 - \frac{m}{15} = 150e^{-\frac{1}{15}t}$$ $$\frac{m}{15} = 150\left(1 - e^{-\frac{1}{15}t}\right)$$ $$m = 2250\left(1 - e^{-\frac{1}{15}t}\right)$$ Find the mass (in grams) of pollutant in the tank after $t =$ 10 days: $$m = 2250\left(1 - e^{-\frac{1}{15}t}\right) = 2250\left(1 - e^{-\frac{2}{3}}\right)$$ Since the volume of the pond is constant at $V = 450$ litres, the concentration of the pollutant in the pond after $t = 10$ days is: $$\frac{m}{V} = \frac{2250\left(1 - e^{-\frac{2}{3}}\right)}{450} = 5\left(1 - e^{-\frac{2}{3}}\right) \approx 2.43\ g/L$$	3	**3 for correct solution.** **2 for correctly substituting the initial condition.** **1 for correctly integrating.**

Markers Comments

Poorly done by almost all the cohort. Very concerning issues involving integration of logs and splitting up the DE to integrate were present.